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# Efficiency Conditions and Duality for a Class of Multiobjective Fractional Programming Problems

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**Abstract.** A class of constrained multiobjective fractional programming problems is considered from a viewpoint of the generalized convexity. Some basic concepts about the generalized convexity of functions, including a unified formulation of generalized convexity, are presented. Based upon the concept of the generalized convexity, efficiency conditions and duality for a class of multiobjective fractional programming problems are obtained. For three types of duals of the multiobjective fractional programming problem, the corresponding duality theorems are also established.

**Key words:** duality, efficiency condition, efficient solution, $(F, \alpha, \rho, d)$-convex functions, Multiobjective fractional programming problem

## 1. Introduction

A number of optimization problems are actually multiobjective optimization problems (MOPs), where the objectives are conflicting. As a result, there is usually no single solution which optimizes all objectives simultaneously. A number of techniques have been developed to find a compromise solution to MOPs. The reader is referred to the recent book by Miettinen [16] about the theory and algorithms for MOPs. Fractional programming problems(FPPs) arise from many applied areas such as portfolio selection, stock cutting, game theory, and numerous decision problems in management science. Many approaches for FPPs have been exploited in considerable details. See, for example, Avriel et al. [3], Craven [5], Schaible [24, 25], Schaible and Ibaraki [26] and Stancu-Minasian [27, 28].

In this paper, we consider the following multiobjective fractional programming problem:

$$\begin{aligned}(\text{MFP})\quad &\min\ \frac{f(x)}{g(x)} \triangleq \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\right)^T,\\ &\text{s.t.}\quad h(x) \leqslant 0, x \in X,\end{aligned}$$

where $X \subset R^n$ is an open set, $f_i, g_i$ $(i = 1, 2, \cdots, p)$ are real-valued functions defined on $X$, and $h$ is an $m$-dimensional vector-valued function defined on $X$. Suppose that $f_i(x) \geqslant 0$ and $g_i(x) > 0$ for $x \in X$ and $i = 1, 2, \cdots, p$. Moreover, let $f_i, g_i$ $(i = 1, 2, \cdots, p)$ and $h_j$ $(j = 1, 2, \cdots, m)$ be continuously differentiable over $X$ and denote the gradients of $f_i, g_i$ and $h_j$ at $x$ by $\nabla f_i(x)$, $\nabla g_i(x)$ and $\nabla h_j(x)$, respectively.

If the parameter $p$ in the problem (MFP) is equal to 1, then (MFP) corresponds to the following single-objective fractional programming problem:

$$\text{(FP)} \quad \min \ \frac{f(x)}{g(x)}, \qquad \text{s.t.} \quad h(x) \leqslant 0, x \in X,$$

where $X \subset R^n$ is an open set, $f, g$ are real-valued functions defined on $X$, and $h$ is an $m$-dimensional vector-valued function defined on $X$, $f(x) \geqslant 0$ and $g(x) > 0$ for all $x \in X$. Moreover, assume that $f(x)$, $g(x)$ and $h_j(x)$ $(j = 1, 2, \cdots, m)$ are continuously differentiable over $X$.

Khan and Hanson [10], and Reddy and Mukherjee [21] considered the optimality conditions and duality for (FP) with respect to the following generalized concepts of convexity, respectively.

DEFINITION 1.1. *[6]* Let $f$ be a real function defined on an open set $X \subseteq R^n$ and differentiable at $x_0$. Given a mapping $\eta : X \times X \to R^n$, the function $f$ is said to be invex at $x_0$ with respect to $\eta$ if, $\forall x \in X$, the following inequality holds:

$$f(x) - f(x_0) \geqslant \nabla f(x_0)^T \eta(x, x_0).$$

DEFINITION 1.2. *[7]* Let $f$ be a real function defined on an open set $X \subseteq R^n$ and differentiable at $x_0$. Given a real number $\rho$, a mapping $\eta : X \times X \to R^n$ and a scalar function $d : X \times X \to R$, the function $f$ is said to be $\rho$-invex at $x_0$ with respect to $\eta$ and $d$ if, $\forall x \in X$, the following inequality holds:

$$f(x) - f(x_0) \geqslant \nabla f(x_0)^T \eta(x, x_0) + \rho d^2(x, x_0).$$

The authors of references [10, 21] imposed the corresponding generalized convexity on the numerator and denominator individually for the objective function in the problem (FP), and then derived some optimality conditions and duality results. How to extend these methods to the multiobjective case is still an open problem [21].

As far as the multiobjective fractional problem (MFP) is concerned, Jeyakumar and Mond [8] introduced a concept of $v$-invexity as follows.

DEFINITION 1.3. Let $f : X \to R^p$ be a real vector function defined on an open set $X \subseteq R^n$ and each component of $f$ be differentiable at $x_0$. The function $f$ is said to be $v$-invex at $x_0 \in X$ if there exist a mapping $\eta : X \times X \to R^n$ and a function $\alpha_i : X \times X \to R_+ \setminus \{0\}$ $(i = 1, 2, \cdots, p)$ such that, $\forall x \in X$,

$$f_i(x) - f_i(x_0) \geqslant \alpha_i(x, x_0) \nabla f_i(x_0)^T \eta(x, x_0).$$

Jeyakumar and Mond [8] obtained some weak efficiency conditions and duality results for a nonconvex multiobjective fractional programming problem via the concept of $v$-invexity, $v$-pseudoinvexity and $v$-quasiinvexity.

Motivated by various concepts of generalized convexity, Liang et al. [12] introduced a unified formulation of the generalized convexity, which was called $(F, \alpha, \rho, d)$-convexity, and obtained some corresponding optimality conditions and duality results for the single-objective fractional problem (FP). In this paper, we will extend the methods adopted for the single-objective problem (FP) in [12] to the multiobjective problem (MFP).

DEFINITION 1.4. A function $F : R^n \to R$ is said to be sublinear if for any $\alpha_1, \alpha_2 \in R^n$,

$$F(\alpha_1 + \alpha_2) \leqslant F(\alpha_1) + F(\alpha_2), \tag{1}$$

and for any $r \in R_+, \alpha \in R^n$,

$$F(r\alpha) = rF(\alpha). \tag{2}$$

Note that the concept of the sublinear function was given in Preda [20]. Now, a sublinear function is defined simply as a function that is subadditive and positively homogeneous, which is free of extraneous symbols in Preda [20]. It follows from (2) that $F(0) = 0$.

Based upon the concept of the sublinear function, we recall the unified formulation about generalized convexity, i.e., $(F, \alpha, \rho, d)$-convexity, which was introduced in [12] as follows.

DEFINITION 1.5. Given an open set $X \subset \Re^n$, a number $\rho \in R$, and two functions $\alpha : X \times X \to R_+ \setminus \{0\}$ and $d : X \times X \to R$, a differentiable function $f$ over $X$ is said to be $(F, \alpha, \rho, d)$-convex at $x_0 \in X$ if for any $x \in X$, $F(x, x_0; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is sublinear, and $f(x)$ satisfies the following condition:

$$f(x) - f(x_0) \geqslant F(x, x_0; \alpha(x, x_0)\nabla f(x_0)) + \rho d^2(x, x_0). \tag{3}$$

The function $f$ is said to be $(F, \alpha, \rho, d)$-convex over $X$ if, $\forall x_0 \in X$, it is $(F, \alpha, \rho, d)$-convex at $x_0$; $f$ is said to be strongly $(F, \alpha, \rho, d) - convex$ or $(F, \alpha) - convex$ if $\rho > 0$ or $\rho = 0$, respectively.

From Definition 1.5, there are the following special cases:

(i) If $\alpha(x, x_0) = 1$ for all $x, x_0 \in X$, then $(F, \alpha, \rho, d)$-convexity is $(F, \rho)$-convexity [20].
(ii) If $F(x, x_0; \alpha(x, x_0)\nabla f(x_0)) = \nabla f(x_0)^T \eta(x, x_0)$ for a certain mapping $\eta : X \times X \to R^n$, then $(F, \alpha, \rho, d)$-convexity is $\rho$-invexity defined in [7].

(iii) If $\rho = 0$ or $d(x, x_0) \equiv 0$ for all $x, x_0 \in X$ and $F(x, x_0; \alpha(x, x_0)\, \nabla f(x_0)) = \nabla f(x_0)^T \eta(x, x_0)$ for a certain mapping $\eta : X \times X \to R^n$, then $(F, \alpha, \rho, d)$-convexity reduces to invexity [6].

In the following, $\rho$, $\alpha$ and $d$ are referred to as parameters of $(F, \alpha, \rho, d)$-convexity. Furthermore, we will adopt the following conventions.

Let $R^n_+$ denote the nonnegative orthant of $R^n$ and $x^T$ denote the transpose of the vector $x \in R^n$. For any two vectors $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T \in R^n$, we denote:

$$x = y \text{ implying } x_i = y_i,\ i = 1, 2, \cdots, n;$$
$$x \prec y \text{ implying } x_i \leqslant y_i,\ i = 1, 2, \cdots, n, \text{ but } x \neq y;$$
$$x < y \text{ implying } x_i < y_i,\ i = 1, 2, \cdots, n;$$
$$x \not< y \text{ implying } y_i < x_i \text{ for at least one } i.$$

A solution of the problem (MFP) is referred to as an efficient (Pareto optimal) solution, which is defined as follows.

DEFINITION 1.6. A feasible solution $x_0 \in X$ of (MFP) is called an efficient solution of (MFP) if there exists no other feasible solution $x \in X$ such that

$$\frac{f(x)}{g(x)} \prec \frac{f(x_0)}{g(x_0)}.$$

In [14], Maeda gave a kind of constraint qualification, which was called generalized Guignard constraint qualification(GGCQ), under which he derived the following Kuhn–Tucker type necessary conditions for a feasible solution $x_0$ to be an efficient solution to the problem (MFP):

If $x_0$ is an efficient solution of (MFP) and (GGCQ) holds at $x_0$ (Ref. [14]), then there exist
$\tau = (\tau_1, \tau_2, \cdots, \tau_p)^T \in R^p_+$, $\tau > 0$, $\sum_{i=1}^{p} \tau_i = 1$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T \in R^m_+$
such that

$$\sum_{i=1}^{p} \tau_i \nabla \frac{f_i(x_0)}{g_i(x_0)} + \sum_{j=1}^{m} \lambda_j \nabla h_j(x_0) = 0,$$
$$\lambda_j h_j(x_0) = 0, \qquad j = 1, 2, \cdots, m.$$

This paper is organized as follows. In Section 2, efficiency conditions for the multiobjective fractional problem (MFP) involving $(F, \alpha, \rho, d)$-convexity are presented. The duality properties of the problem (MFP) are studied in Section 3, including several duals for (MFP) and some weak and strong duality theorems. Concluding remarks are given in the last section.

## 2. Efficiency Conditions

First, we present a lemma which indicates that $(F, \alpha, \rho, d)$-convexity can be preserved after taking division.

LEMMA 2.1. *Let $X \subset R^n$ be an open set. Assume that $p$, $q$ are real-valued differentiable functions defined on $X$ and $p(x) \geqslant 0, q(x) > 0$ for all $x \in X$. If $p$ and $-q$ are $(F, \alpha, \rho, d)$-convex at $x_0 \in X$, then $\frac{p}{q}$ is $(F, \overline{\alpha}, \overline{\rho}, \overline{d})$-convex at $x_0$, where*

$$\overline{\alpha}(x, x_0) = \frac{\alpha(x, x_0)q(x_0)}{q(x)}, \overline{\rho} = \rho\left(1 + \frac{p(x_0)}{q(x_0)}\right), \text{ and } \overline{d}(x, x_0) = \frac{d(x, x_0)}{q^{\frac{1}{2}}(x)}.$$

*Proof.* For any $x \in X$, we have

$$\frac{p(x)}{q(x)} - \frac{p(x_0)}{q(x_0)} = \frac{p(x) - p(x_0)}{q(x)} - \frac{p(x_0)(q(x) - q(x_0))}{q(x)q(x_0)}.$$

By the $(F, \alpha, \rho, d)$-convexity of $p$ and $-q$, and $p \geqslant 0, q > 0$, we obtain

$$\begin{aligned}\frac{p(x)}{q(x)} - \frac{p(x_0)}{q(x_0)} \geqslant\ & \frac{1}{q(x)}\Big(F\big(x, x_0; \alpha(x, x_0)\nabla p(x_0)\big) + \rho d^2(x, x_0)\Big) \\ & + \frac{p(x_0)}{q(x)q(x_0)}\Big(F(x, x_0; -\alpha(x, x_0)\nabla q(x_0)) + \rho d^2(x, x_0)\Big).\end{aligned}$$

Based on the sublinearity of $F$ and $p \geqslant 0, q > 0$, the following inequalities can be obtained:

$$\begin{aligned}\frac{p(x)}{q(x)} - \frac{p(x_0)}{q(x_0)} \geqslant\ & F\left(x, x_0; \frac{\alpha(x, x_0)}{q(x)}\nabla p(x_0)\right) + \rho\frac{d^2(x, x_0)}{q(x)} \\ & + F\left(x, x_0; -\frac{\alpha(x, x_0)p(x_0)}{q(x)q(x_0)}\nabla q(x_0)\right) + \rho\frac{d^2(x, x_0)p(x_0)}{q(x)q(x_0)} \\ \geqslant\ & F\left(x, x_0; \frac{\alpha(x, x_0)}{q(x)} \cdot \frac{q(x_0)\nabla p(x_0) - p(x_0)\nabla q(x_0)}{q(x_0)}\right) \\ & + \rho\left(1 + \frac{p(x_0)}{q(x_0)}\right)\frac{d^2(x, x_0)}{q(x)} \\ =\ & F\left(x, x_0; \frac{\alpha(x, x_0)q(x_0)}{q(x)}\nabla\frac{p(x_0)}{q(x_0)}\right) \\ & + \rho\left(1 + \frac{p(x_0)}{q(x_0)}\right)\frac{d^2(x, x_0)}{q(x)}.\end{aligned}$$

Denote

$$\overline{\alpha}(x, x_0) = \frac{\alpha(x, x_0)q(x_0)}{q(x)}, \overline{\rho} = \rho\left(1 + \frac{p(x_0)}{q(x_0)}\right), \text{ and } \overline{d}(x, x_0) = \frac{d(x, x_0)}{q^{\frac{1}{2}}(x)}.$$

Then we have

$$\frac{p(x)}{q(x)} - \frac{p(x_0)}{q(x_0)} \geqslant F\left(x, x_0; \overline{\alpha}(x, x_0)\nabla\frac{p(x_0)}{q(x_0)}\right) + \overline{\rho}\overline{d}^2(x, x_0) \quad \forall x \in X.$$

Therefore, $\frac{p}{q}$ is $(F, \overline{\alpha}, \overline{\rho}, \overline{d})$-convex at $x_0$. □

In the following, we present some sufficient efficiency conditions for (MFP) under appropriate $(F, \alpha, \rho, d)$-convexity assumptions.

THEOREM 2.1. *Let $x_0$ be a feasible solution of (MFP). Suppose that there exist $\tau = (\tau_1, \tau_2, \cdots, \tau_p)^T \in R_+^p$, $\tau > 0$, $\sum_{i=1}^{p} \tau_i = 1$, and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T \in R_+^m$ such that*

$$\sum_{i=1}^{p} \tau_i \nabla \frac{f_i(x_0)}{g_i(x_0)} + \sum_{j=1}^{m} \lambda_j \nabla h_j(x_0) = 0, \tag{4}$$

$$\lambda_j h_j(x_0) = 0, \qquad j = 1, 2, \cdots, m. \tag{5}$$

*If $f_i$ and $-g_i$ $(i = 1, 2, \cdots, p)$ are $(F, \alpha_i, \rho_i, d_i)$-convex at $x_0$, $h_j$ $(j = 1, 2, \cdots, m)$ is $(F, \beta_j, \zeta_j, c_j)$-convex at $x_0$, and*

$$\sum_{i=1}^{p} \tau_i \overline{\rho}_i \frac{\overline{d}_i^2(x, x_0)}{\overline{\alpha}_i(x, x_0)} + \sum_{j=1}^{m} \lambda_j \zeta_j \frac{c_j^2(x, x_0)}{\beta_j(x, x_0)} \geqslant 0, \tag{6}$$

*where* $\overline{\alpha}_i(x, x_0) = \frac{\alpha_i(x, x_0) g_i(x_0)}{g_i(x)}$, $\overline{\rho}_i = \rho_i \left(1 + \frac{f_i(x_0)}{g_i(x_0)}\right)$, *and* $\overline{d}_i(x, x_0) = \frac{d_i(x, x_0)}{g_i^{\frac{1}{2}}(x)}$, *then $x_0$ is a global efficient solution for (MFP).*

*Proof.* Suppose that $x_0$ is not a global efficient solution of (MFP). Then there exists a feasible solution $x$ such that

$$\frac{f(x)}{g(x)} \prec \frac{f(x_0)}{g(x_0)},$$

that is, for $i = 1, 2, \cdots, p$,

$$\frac{f_i(x)}{g_i(x)} \leqslant \frac{f_i(x_0)}{g_i(x_0)}$$

and at least one inequality holds strictly.

By Lemma 2.1, for each $i$, $1 \leqslant i \leqslant p$, $\frac{f_i}{g_i}$ is $(F, \overline{\alpha}_i, \overline{\rho}_i, \overline{d}_i)$-convex, i.e.,

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(x_0)}{g_i(x_0)} \geqslant F\left(x, x_0; \overline{\alpha}_i(x, x_0) \nabla \frac{f_i(x_0)}{g_i(x_0)}\right) + \overline{\rho}_i \overline{d}_i^2(x, x_0),$$

where

$$\overline{\alpha}_i(x, x_0) = \frac{\alpha_i(x, x_0) g_i(x_0)}{g_i(x)}, \overline{\rho}_i = \rho_i \left(1 + \frac{f_i(x_0)}{g_i(x_0)}\right), \text{ and } \overline{d}_i(x, x_0) = \frac{d_i(x, x_0)}{g_i^{\frac{1}{2}}(x)}.$$

Since $\overline{\alpha}_i(x, x_0) > 0$, by the sublinearity of $F$, we have

$$\frac{1}{\overline{\alpha}_i(x, x_0)} \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(x_0)}{g_i(x_0)}\right) \geqslant F\left(x, x_0; \nabla \frac{f_i(x_0)}{g_i(x_0)}\right) + \overline{\rho}_i \frac{\overline{d}_i^2(x, x_0)}{\overline{\alpha}_i(x, x_0)},$$

where the left-hand side of the above inequality is less than or equal to zero. Hence, we obtain the following $p$ inequalities,

$$F\left(x, x_0; \nabla \frac{f_i(x_0)}{g_i(x_0)}\right) + \overline{\rho}_i \frac{\overline{d}_i^2(x, x_0)}{\overline{\alpha}_i(x, x_0)} \leqslant 0, i = 1, 2, \cdots, p,$$

and at least one inequality holds strictly.

Multiplying the above $p$ inequalities with $\tau_i$, respectively, and then adding them together, we have

$$\sum_{i=1}^{p} \tau_i F\left(x, x_0; \nabla \frac{f_i(x_0)}{g_i(x_0)}\right) + \sum_{i=1}^{p} \tau_i \overline{\rho}_i \frac{\overline{d}_i^2(x, x_0)}{\overline{\alpha}_i(x, x_0)} < 0.$$

By the sublinearity of $F$ and $\tau_i > 0$ $(i = 1, 2, \cdots, p)$, we know that

$$\sum_{i=1}^{p} \tau_i F\left(x, x_0; \nabla \frac{f_i(x_0)}{g_i(x_0)}\right) \geqslant F\left(x, x_0; \sum_{i=1}^{p} \tau_i \nabla \frac{f_i(x_0)}{g_i(x_0)}\right).$$

Hence, we get

$$F\left(x, x_0; \sum_{i=1}^{p} \tau_i \nabla \frac{f_i(x_0)}{g_i(x_0)}\right) + \sum_{i=1}^{p} \tau_i \overline{\rho}_i \frac{\overline{d}_i^2(x, x_0)}{\overline{\alpha}_i(x, x_0)} < 0. \tag{7}$$

Substituting (4) into (7), we obtain

$$F(x, x_0; -\sum_{j=1}^{m} \lambda_j \nabla h_j(x_0)) + \sum_{i=1}^{p} \tau_i \overline{\rho}_i \frac{\overline{d}_i^2(x, x_0)}{\overline{\alpha}_i(x, x_0)} < 0. \tag{8}$$

The sublinearity of $F$ and (6) yield

$$\begin{aligned}
&F(x, x_0; -\sum_{j=1}^{m} \lambda_j \nabla h_j(x_0)) + \sum_{i=1}^{p} \tau_i \overline{\rho}_i \frac{\overline{d}_i^2(x, x_0)}{\overline{\alpha}_i(x, x_0)} \\
&+F(x, x_0; \sum_{j=1}^{m} \lambda_j \nabla h_j(x_0)) + \sum_{j=1}^{m} \lambda_j \zeta_j \frac{c_j^2(x, x_0)}{\beta_j(x, x_0)} \\
&\geqslant \sum_{i=1}^{p} \tau_i \overline{\rho}_i \frac{\overline{d}_i^2(x, x_0)}{\overline{\alpha}_i(x, x_0)} + \sum_{j=1}^{m} \lambda_j \zeta_j \frac{c_j^2(x, x_0)}{\beta_j(x, x_0)} \\
&\geqslant 0.
\end{aligned}$$

Using (8), we obtain

$$F(x, x_0; \sum_{j=1}^{m} \lambda_j \nabla h_j(x_0)) + \sum_{j=1}^{m} \lambda_j \zeta_j \frac{c_j^2(x, x_0)}{\beta_j(x, x_0)} > 0. \tag{9}$$

On the other hand, for $j = 1, 2, \cdots, m$, by the $(F, \beta_j, \zeta_j, c_j)$-convexity of $h_j$, we have

$$h_j(x) - h_j(x_0) \geqslant F(x, x_0; \beta_j(x, x_0)\nabla h_j(x_0)) + \zeta_j c_j^2(x, x_0).$$

By using $\lambda_j \geqslant 0$, $\beta_j(x, x_0) > 0$ and the sublinearity of $F$, we have

$$\sum_{j=1}^{m} \lambda_j \frac{h_j(x) - h_j(x_0)}{\beta_j(x, x_0)} \geqslant F(x, x_0; \sum_{j=1}^{m} \lambda_j \nabla h_j(x_0)) + \sum_{j=1}^{m} \lambda_j \zeta_j \frac{c_j^2(x, x_0)}{\beta_j(x, x_0)}.$$

Since $x$ is feasible and $\beta_j(x, x_0) > 0$, (5) implies that

$$\sum_{j=1}^{m} \lambda_j \frac{h_j(x) - h_j(x_0)}{\beta_j(x, x_0)} \leqslant 0.$$

Then, we obtain

$$F(x, x_0; \sum_{j=1}^{m} \lambda_j \nabla h_j(x_0)) + \sum_{j=1}^{m} \lambda_j \zeta_j \frac{c_j^2(x, x_0)}{\beta_j(x, x_0)} \leqslant 0, \tag{10}$$

which contradicts (9). Therefore, $x_0$ is a global efficient solution for (MFP). □

COROLLARY 2.1. *Let $x_0$ be a feasible solution of (MFP). Suppose that there exist $\tau = (\tau_1, \tau_2, \cdots, \tau_p)^T \in R_+^p$, $\tau > 0$, $\sum_{i=1}^{p} \tau_i = 1$, and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T \in R_+^m$ such that*

$$\sum_{i=1}^{p} \tau_i \nabla \frac{f_i(x_0)}{g_i(x_0)} + \sum_{j=1}^{m} \lambda_j \nabla h_j(x_0) = 0,$$
$$\lambda_j h_j(x_0) = 0, \qquad j = 1, 2, \cdots, m.$$

*If $f_i$ and $-g_i (i = 1, 2, \cdots, p)$ are strongly $(F, \alpha_i, \rho_i, d_i)$-convex (or $(F, \alpha_i)$-convex) at $x_0$, $h_j (j = 1, 2, \cdots, m)$ is strongly $(F, \beta_j, \zeta_j, c_j)$-convex (or $(F, \beta_j)$-convex) at $x_0$, then $x_0$ is a global efficient solution for (MFP).*

*Proof.* We use the same notations as those in Theorem 2.1. Since $\overline{\rho}_i = \rho_i\left(1 + \frac{f_i(x_0)}{g_i(x_0)}\right)$ and $f_i(x_0) \geqslant 0$, $g_i(x_0) > 0$, $i = 1, 2, \cdots, p$, under the assumptions of this corollary, we know that the inequality (6) holds. Therefore, $x_0$ is a global efficient solution of (MFP). □

For $i = 1, 2, \cdots, p$, if $g_i(x) = 1$ for all $x \in X$, $f_i(x)$ need not be nonnegative, and the functions involved are assumed to be invex, $\rho-$invex with respect to $\eta : X \times X \to R^n$, $d : X \times X \to R$, $(F, \rho)$-convex, or generalized $(F, \rho)$-convex, respectively, then we can obtain the corresponding results presented in [1, 2, 9].

Next, we consider a special case of (MFP), in which the fractional objective functions have the same denominator. For $i = 1, 2, \cdots, p$, let $g_i(x) = g(x)$ in (MFP). The property about the efficient solution of this special (MFP) can be obtained similarly as that in Theorem 2.1, so we state the following theorem:

THEOREM 2.2. *Let $x_0$ be a feasible solution of (MFP). Suppose that there exist $\tau = (\tau_1, \tau_2, \cdots, \tau_p)^T \in R_+^p$, $\tau > 0$, $\sum_{i=1}^{p} \tau_i = 1$, and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T \in R_+^m$ such that*

$$\sum_{i=1}^{p} \tau_i \nabla \frac{f_i(x_0)}{g(x_0)} + \sum_{j=1}^{m} \lambda_j \nabla h_j(x_0) = 0,$$
$$\lambda_j h_j(x_0) = 0, \qquad j = 1, 2, \cdots, m.$$

*If $-g$ is $(F, \alpha, \rho, d)$-convex at $x_0$, $f_i$ $(i = 1, 2, \cdots, p)$ is $(F, \alpha, \rho_i, d)$-convex at $x_0$, $h_j (j = 1, 2, \cdots, m)$ is $(F, \overline{\alpha}, \zeta_j, \overline{d})$-convex at $x_0$, and $\sum_{i=1}^{p} \tau_i \overline{\rho}_i + \sum_{j=1}^{m} \lambda_j \zeta_j \geqslant 0$, where $\overline{\alpha}(x, x_0) = \frac{\alpha(x, x_0) g(x_0)}{g(x)}$, $\overline{\rho}_i = \rho_i + \rho \frac{f_i(x_0)}{g(x_0)}$, and $\overline{d}(x, x_0) = \frac{d(x, x_0)}{g^{\frac{1}{2}}(x)}$, then $x_0$ is a global efficient solution for (MFP).*

Finally, we present an equivalent formulation of the problem (MFP). Let $G(x) = \prod_{i=1}^{p} g_i(x)$, $G_i(x) = \frac{G(x)}{g_i(x)}$ $(i = 1, 2, \cdots, p)$. Then (MFP) can be written in the following form:

$$(\overline{MFP}) \quad \min \ \frac{f(x)}{g(x)} = \left( \frac{G_1(x) f_1(x)}{G(x)}, \frac{G_2(x) f_2(x)}{G(x)}, \cdots, \frac{G_p(x) f_p(x)}{G(x)} \right)^T,$$
$$\text{s.t.} \quad h(x) \leqslant 0, x \in X.$$

By Theorem 2.2, we have the following corollary:

COROLLARY 2.2. *Let $x_0$ be a feasible solution of (MFP). Suppose that there exist $\tau = (\tau_1, \tau_2, \cdots, \tau_p)^T \in R_+^p$, $\tau > 0$, $\sum_{i=1}^{p} \tau_i = 1$, and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T \in R_+^m$ such that*

$$\sum_{i=1}^{p} \tau_i \nabla \frac{f_i(x_0)}{g_i(x_0)} + \sum_{j=1}^{m} \lambda_j \nabla h_j(x_0) = 0,$$
$$\lambda_j h_j(x_0) = 0, \ j = 1, 2, \cdots, m.$$

*If $-G$ is $(F, \alpha, \rho, d)$-convex at $x_0$, $G_i f_i$ $(i = 1, 2, \cdots, p)$ is $(F, \alpha, \rho_i, d)$-convex at $x_0$, $h_j$ $(j = 1, 2, \cdots, m)$ is $(F, \overline{\alpha}, \zeta_j, \overline{d})$-convex at $x_0$, and $\sum_{i=1}^{p} \tau_i \overline{\rho}_i + \sum_{j=1}^{m} \lambda_j \zeta_j \geqslant 0$, where $\overline{\rho}_i = \rho_i + \rho \frac{f_i(x_0)}{g_i(x_0)}$, $\overline{\alpha}(x, x_0) = \frac{\alpha(x, x_0) G(x_0)}{G(x)}$, and $\overline{d}(x, x_0) = \frac{d(x, x_0)}{G^{\frac{1}{2}}(x)}$, then $x_0$ is a global efficient solution for (MFP).*

Under the assumptions of Theorem 2.2 or Corollary 2.2, if $\rho \geqslant \max_{1 \leqslant i \leqslant p} \rho_i$, $\overline{\rho}_i = \rho_i\left(1 + \frac{f_i(x_0)}{g(x_0)}\right)$, or $\overline{\rho}_i = \rho_i\left(1 + \frac{f_i(x_0)}{g_i(x_0)}\right)$, respectively, then the corresponding results still hold.

## 3. Duality

In optimization theory, there are many types of duals for a given mathematical programming problem. Two well-known duals are the Wolfe type dual [29] and the Mond-Weir type dual [17]. Recently, the mixed (or general type) dual has been considered for various optimization problems [1, 2, 11, 13, 18, 19, 20, 30, 31, 32]. The mixed dual includes the Wolfe type dual and the Mond-Weir type dual as special cases. In the sequel, the generalized Mond-Weir dual is discussed first, and then three other types of duals are presented, which are based on $(F, \alpha, \rho, d)$-convexity for the problem (MFP).

Let $M = \{1, 2, \cdots, m\}$ and $M_0, M_1, \cdots, M_q$ be a partition of $M$, i.e., $\bigcup_{k=0}^{q} M_k = M$, $M_k \bigcap M_l = \emptyset$ for $k \neq l$. The generalized Mond-Weir dual of (MFP) is as follows:

$$
\begin{aligned}
\max \quad & \frac{f(u)}{g(u)} + \lambda_{M_0}^T h_{M_0}(u)\, e :\stackrel{\Delta}{=} \\
& \left(\frac{f_1(u)}{g_1(u)} + \lambda_{M_0}^T h_{M_0}(u), \cdots, \frac{f_p(u)}{g_p(u)} + \lambda_{M_0}^T h_{M_0}(u)\right)^T, \\
\text{s.t.} \quad & \sum_{i=1}^{p} \tau_i \nabla \frac{f_i(u)}{g_i(u)} + \sum_{j=1}^{m} \lambda_j \nabla h_j(u) = 0, \\
& \lambda_{M_k}^T h_{M_k}(u) \geqslant 0, k = 1, 2, \cdots, q, \\
& \tau = (\tau_1, \tau_2, \cdots, \tau_p)^T \in R_+^p, \tau > 0, \sum_{i=1}^{p} \tau_i = 1, \\
& \lambda_{M_k} \in R_+^{|M_k|}, k = 0, 1, 2, \cdots, q, \\
& u \in X,
\end{aligned}
$$

where $e = (1, 1, \cdots, 1)^T$ and $\lambda_{M_k}$ denotes the column vector whose subscripts of components belong to $M_k$. In particular, if $M_0 = M, M_k = \emptyset, k = 1, 2, \cdots, q$, then the above dual becomes the Wolfe type dual; if $M_0 = \emptyset$ and $q = 1, M_1 = M$, the Mond-Weir type dual is obtained. Since the Wolfe type dual is unsuitable for single-objective fractional programming problems [15, 22, 23], the duals with $M_0 \neq \emptyset$ are certainly unsuitable for (MFP). For the generalized Mond-Weir type dual, we only consider the case $M_0 = \emptyset$, $M_1 = M$, i.e., the Mond-Weir dual.

### 3.1. MOND-WEIR DUAL

The Mond-Weir dual of the problem (MFP) has the following form:

$$\text{(MFD1)}\quad \max\ \frac{f(u)}{g(u)} = \left(\frac{f_1(u)}{g_1(u)}, \frac{f_2(u)}{g_2(u)}, \cdots, \frac{f_p(u)}{g_p(u)}\right)^T,$$
$$\text{s.t.}\quad \sum_{i=1}^{p} \tau_i \nabla \frac{f_i(u)}{g_i(u)} + \sum_{j=1}^{m} \lambda_j \nabla h_j(u) = 0,$$
$$\lambda^T h(u) \geqslant 0,$$
$$\tau = (\tau_1, \tau_2, \cdots, \tau_p)^T \in R^p, \tau > 0, \sum_{i=1}^{p} \tau_i = 1,$$
$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T \in R_+^m, u \in X.$$

THEOREM 3.1. *(Weak Duality) Assume that $\bar{x}$ is a feasible solution of (MFP) and $(\bar{u}, \bar{\tau}, \bar{\lambda})$ is a feasible solution of (MFD1). If $f_i$ and $-g_i$ $(i = 1, 2, \cdots, p)$ are $(F, \alpha_i, \rho_i, d_i)$-convex at $\bar{u}$, $h_j$ $(j = 1, 2, \cdots, m)$ is $(F, \beta, \zeta_j, c_j)$-convex at $\bar{u}$, and the inequality*

$$\sum_{i=1}^{p} \bar{\tau}_i \bar{\rho}_i \frac{\bar{d}_i^2(\bar{x}, \bar{u})}{\bar{\alpha}_i(\bar{x}, \bar{u})} + \sum_{j=1}^{m} \bar{\lambda}_j \zeta_j \frac{c_j^2(\bar{x}, \bar{u})}{\beta(\bar{x}, \bar{u})} \geqslant 0 \tag{11}$$

*holds, where $\bar{\alpha}_i(\bar{x}, \bar{u}) = \alpha_i(\bar{x}, \bar{u}) \frac{g(\bar{u})}{g(\bar{x})}$, $\bar{\rho}_i = \rho_i\left(1 + \frac{f_i(\bar{u})}{g_i(\bar{u})}\right)$, and $\bar{d}_i(\bar{x}, \bar{u}) = \frac{d_i(\bar{x}, \bar{u})}{g_i^{\frac{1}{2}}(\bar{x})}$, then we have*

$$\frac{f(\bar{x})}{g(\bar{x})} \not\prec \frac{f(\bar{u})}{g(\bar{u})}.$$

*Proof.* Suppose that

$$\frac{f(\bar{x})}{g(\bar{x})} \prec \frac{f(\bar{u})}{g(\bar{u})},$$

that is,

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} - \frac{f_i(\bar{u})}{g_i(\bar{u})} \leqslant 0, i = 1, 2, \cdots, p, \tag{12}$$

and at least one inequality holds strictly.

For each $i$, $1 \leqslant i \leqslant p$, by the generalized convexity assumptions and Lemma 2.1, we have

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} - \frac{f_i(\bar{u})}{g_i(\bar{u})} \geqslant F\left(\bar{x}, \bar{u}; \bar{\alpha}_i(\bar{x}, \bar{u}) \nabla \frac{f_i(\bar{u})}{g_i(\bar{u})}\right) + \bar{\rho}_i \bar{d}_i^2(\bar{x}, \bar{u}),$$

where $\bar{\alpha}_i(\bar{x}, \bar{u}) = \alpha_i(\bar{x}, \bar{u}) \frac{g(\bar{u})}{g(\bar{x})}$, $\bar{\rho}_i = \rho_i\left(1 + \frac{f_i(\bar{u})}{g_i(\bar{u})}\right)$, and $\bar{d}_i(\bar{x}, \bar{u}) = \frac{d_i(\bar{x}, \bar{u})}{g_i^{\frac{1}{2}}(\bar{x})}$.

Using $\bar{\tau}_i > 0$, $\bar{\alpha}_i(\bar{x}, \bar{u}) > 0$ and (2), we get

$$\frac{\bar{\tau}_i}{\bar{\alpha}_i(\bar{x}, \bar{u})}\left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \frac{f_i(\bar{u})}{g_i(\bar{u})}\right) \geqslant F\left(\bar{x}, \bar{u}; \bar{\tau}_i \nabla \frac{f_i(\bar{u})}{g_i(\bar{u})}\right) + \bar{\tau}_i \bar{\rho}_i \frac{\bar{d}_i^2(\bar{x}, \bar{u})}{\bar{\alpha}_i(\bar{x}, \bar{u})}.$$

Then, by (12), we obtain

$$F\left(\overline{x}, \overline{u}; \overline{\tau}_i \nabla \frac{f_i(\overline{u})}{g_i(\overline{u})}\right) + \overline{\tau}_i \overline{\rho}_i \frac{\overline{d}_i^2(\overline{x}, \overline{u})}{\overline{\alpha}_i(\overline{x}, \overline{u})} \leqslant 0, i = 1, 2, \cdots, p.$$

Furthermore, at least one of the above inequalities holds strictly. After adding these inequalities together, we get

$$\sum_{i=1}^{p} F\left(\overline{x}, \overline{u}; \overline{\tau}_i \nabla \frac{f_i(\overline{u})}{g_i(\overline{u})}\right) + \sum_{i=1}^{p} \overline{\tau}_i \overline{\rho}_i \frac{\overline{d}_i^2(\overline{x}, \overline{u})}{\overline{\alpha}_i(\overline{x}, \overline{u})} < 0.$$

Hence, it follows from (1) that

$$F\left(\overline{x}, \overline{u}; \sum_{i=1}^{p} \overline{\tau}_i \nabla \frac{f_i(\overline{u})}{g_i(\overline{u})}\right) + \sum_{i=1}^{p} \overline{\tau}_i \overline{\rho}_i \frac{\overline{d}_i^2(\overline{x}, \overline{u})}{\overline{\alpha}_i(\overline{x}, \overline{u})} < 0. \tag{13}$$

By the $(F, \beta, \zeta_j, c_j)$-convexity of $h_j$, $j = 1, 2, \cdots, m$, we have

$$h_j(\overline{x}) - h_j(\overline{u}) \geqslant F(\overline{x}, \overline{u}; \beta(\overline{x}, \overline{u}) \nabla h_j(\overline{u})) + \zeta_j c_j^2(\overline{x}, \overline{u}).$$

Using $\overline{\lambda}_j \geqslant 0$ and $\beta(\overline{x}, \overline{u}) > 0$, we get

$$\overline{\lambda}_j \frac{h_j(\overline{x}) - h_j(\overline{u})}{\beta(\overline{x}, \overline{u})} \geqslant F(\overline{x}, \overline{u}; \overline{\lambda}_j \nabla h_j(\overline{u})) + \overline{\lambda}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})}, j = 1, 2, \cdots, m.$$

Adding these inequalities together and using the feasibility of $\overline{x}$ and $(\overline{u}, \overline{\tau}, \overline{\lambda})$, we obtain

$$\sum_{j=1}^{m} F(\overline{x}, \overline{u}; \overline{\lambda}_j \nabla h_j(\overline{u})) + \sum_{j=1}^{m} \overline{\lambda}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})} \leqslant 0.$$

Using (1) again, we have

$$F(\overline{x}, \overline{u}; \sum_{j=1}^{m} \overline{\lambda}_j \nabla h_j(\overline{u})) + \sum_{j=1}^{m} \overline{\lambda}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})} \leqslant 0. \tag{14}$$

Based on the sublinearity of $F$, the constraints of (MFD1), (11), (13) and (14), the following contradiction occurs:

$$\begin{aligned} 0 = F(\overline{x}, \overline{u}; 0) &= F\left(\overline{x}, \overline{u}; \sum_{i=1}^{p} \overline{\tau}_i \nabla \frac{f_i(\overline{u})}{g_i(\overline{u})} + \sum_{j=1}^{m} \overline{\lambda}_j \nabla h_j(\overline{u})\right) \\ &\leqslant F\left(\overline{x}, \overline{u}; \sum_{i=1}^{p} \overline{\tau}_i \nabla \frac{f_i(\overline{u})}{g_i(\overline{u})}\right) + F(\overline{x}, \overline{u}; \sum_{j=1}^{m} \overline{\lambda}_j \nabla h_j(\overline{u})) \\ &< -\left(\sum_{i=1}^{p} \overline{\tau}_i \overline{\rho}_i \frac{\overline{d}_i^2(\overline{x}, \overline{u})}{\overline{\alpha}_i(\overline{x}, \overline{u})} + \sum_{j=1}^{m} \overline{\lambda}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})}\right) \\ &\leqslant 0. \end{aligned}$$

Therefore, if follows that

$$\frac{f(\overline{x})}{g(\overline{x})} \not\prec \frac{f(\overline{u})}{g(\overline{u})}.$$

□

COROLLARY 3.1. *(Weak Duality) Assume that $\overline{x}$ is a feasible solution of (MFP), and $(\overline{u}, \overline{\tau}, \overline{\lambda})$ is a feasible solution of (MFD1). If $f_i$ and $-g_i (i = 1, 2, \cdots, p)$ are strongly $(F, \alpha_i, \rho_i, d_i)$-convex (or $(F, \alpha_i)$-convex) at $\overline{u}$, and $h_j(j = 1, 2, \cdots, m)$ is strongly $(F, \beta, \zeta_j, c_j)$-convex (or $(F, \beta)$-convex) at $\overline{u}$, then*

$$\frac{f(\overline{x})}{g(\overline{x})} \not\prec \frac{f(\overline{u})}{g(\overline{u})}.$$

*Proof.* If the conditions of the corollary are satisfied, then $\overline{\rho}_i = \rho_i \left(1+\frac{f_i(u)}{g_i(u)}\right) \geqslant 0, \zeta_j \geqslant 0$. The inequality (11) in Theorem 3.1,

$$\sum_{i=1}^{p} \overline{\tau}_i \overline{\rho}_i \frac{\overline{d}_i^2(\overline{x}, \overline{u})}{\overline{\alpha}_i(\overline{x}, \overline{u})} + \sum_{j=1}^{m} \overline{\lambda}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})} \geqslant 0,$$

holds since all terms in the expression are nonnegative. Hence, the conclusion of this corollary holds. □

THEOREM 3.2. *(Strong Duality) Assume that $\overline{x}$ is an efficient solution of (MFP) and the constraint qualification (GGCQ) holds at $\overline{x}$ (Ref. [14]). Then there exists $(\overline{\tau}, \overline{\lambda}) \in R_+^p \times R_+^m$ such that $(\overline{x}, \overline{\tau}, \overline{\lambda})$ is a feasible solution of (MFD1), and the objective function values of (MFP) and (MFD1) at the corresponding points are equal. If the assumptions about the generalized convexity and the inequality (11) in Theorem 3.1 are also satisfied, then $(\overline{x}, \overline{\tau}, \overline{\lambda})$ is an efficient solution of (MFD1).*

*Proof.* Since $\overline{x}$ is an efficient solution of (MFP) and (GGCQ) holds at $\overline{x}$, by the necessary efficiency conditions, there exists $(\overline{\tau}, \overline{\lambda}) \in R_+^p \times R_+^m, \overline{\tau} > 0$, such that $(\overline{x}, \overline{\tau}, \overline{\lambda})$ is a feasible solution for (MFD1). It is clear that the objective function values of (MFP) and (MFD1) at the corresponding points are equal. If $(\overline{x}, \overline{\tau}, \overline{\lambda})$ is not efficient for (MFD1), then there must exist a feasible solution $(x^*, \tau^*, \lambda^*)$ of (MFD1) such that

$$\frac{f(\overline{x})}{g(\overline{x})} \prec \frac{f(x^*)}{g(x^*)},$$

which contradicts the weak duality result appearing in Theorem 3.1. Therefore, $(\overline{x}, \overline{\tau}, \overline{\lambda})$ is an efficient solution of (MFD1). □

For $i = 1, 2, \cdots, p$, if $g_i(x) = 1, \alpha_i(x, x_0) = 1$ for all $x, x_0 \in X$, the objective function $f_i(x)$ and constraint functions are $(F, \rho)$-convex, then duality results similar to those in [20] can be obtained. Now, let $g_i(x) = 1, \alpha_i(x, x_0) = 1$ for all

$x, x_0 \in X$ and $F(x, x_0; \nabla f(x_0)) = \nabla f(x_0) T_\eta(x, x_0)$. If $\rho_i = 0$ and the modified generalized convexity is imposed on the objective and constraint functions, then the corresponding results as those in [8] can be obtained; if $\rho_i \neq 0$ and other generalized convexity assumptions are imposed on the objective and constraint functions, then results similar to those given in [1, 9] can be obtained.

### 3.2. SCHAIBLE DUAL

In this subsection, we shall consider the following extended form of the Schaible dual for (MFP) [22, 23]:

$$\begin{aligned}
\text{(MFD2)}\quad \max\quad & \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)^T, \\
\text{s.t.}\quad & \sum_{i=1}^{p} \tau_i \nabla_u (f_i(u) - \lambda_i g_i(u)) + \sum_{j=1}^{m} v_j \nabla h_j(u) = 0, \\
& f_i(u) - \lambda_i g_i(u) \geqslant 0, \ i = 1, 2, \cdots, p, \\
& v^T h(u) \geqslant 0, \\
& \tau > 0, \sum_{i=1}^{p} \tau_i = 1, \\
& \lambda \in R_+^p, \tau \in R_+^p, v \in R_+^m, u \in X.
\end{aligned}$$

THEOREM 3.3. *(Weak Duality). Assume that $\overline{x}$ is a feasible solution of (MFP) and $(\overline{u}, \overline{\tau}, \overline{\lambda}, \overline{v})$ is a feasible solution of (MFD2). If one of the following holds:*

(I) *$f_i$ and $-g_i$ $(i = 1, 2, \cdots, p)$ are $(F, \alpha_i, \rho_i, d_i)$-convex at $\overline{u}$ , $h_j$ $(j = 1, 2, \cdots, m)$ is $(F, \beta, \zeta_j, c_j)$-convex at $\overline{u}$, and*

$$\sum_{i=1}^{p} \overline{\tau}_i \rho_i (1 + \overline{\lambda}_i) \frac{d_i^2(\overline{x}, \overline{u})}{\alpha_i(\overline{x}, \overline{u})} + \sum_{j=1}^{m} \overline{v}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})} \geqslant 0; \tag{15}$$

(II) *$f_i$ and $-g_i$ $(i = 1, 2, \cdots, p)$ are $(F, \alpha, \rho_i, d)$-convex at $\overline{u}$, $h_j$ $(j = 1, 2, \cdots, m)$ is $(F, \alpha, \zeta_j, d)$-convex at $\overline{u}$, and the vectors $\overline{\tau}, \overline{\lambda}, \overline{v}$ satisfy:*

$$\sum_{i=1}^{p} \overline{\tau}_i \rho_i (1 + \overline{\lambda}_i) + \sum_{j=1}^{m} \overline{v}_j \zeta_j \geqslant 0, \tag{16}$$

*then*

$$\frac{f(\overline{x})}{g(\overline{x})} \not\prec \overline{\lambda}.$$

*Proof.* Suppose that

$$\frac{f(\overline{x})}{g(\overline{x})} \prec \overline{\lambda},$$

i.e.,

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} \leqslant \overline{\lambda}_i,\ i = 1, 2, \cdots, p,$$

and at least one of the above inequalities holds strictly. Since $g_i(\overline{x}) > 0$ for each $i$, the above inequalities are equivalent to

$$f_i(\overline{x}) \leqslant \overline{\lambda}_i g_i(\overline{x}),\ i = 1, 2, \cdots, p.$$

(I) Since $f_i$ and $-g_i$ are $(F, \alpha_i, \rho_i, d_i)$-convex at $\overline{u}$, using the feasibility of $(\overline{u}, \overline{\tau}, \overline{\lambda}, \overline{v})$ and the sublinearity of $F$, we have

$$\begin{aligned} 0 &\geqslant f_i(\overline{x}) - \overline{\lambda}_i g_i(\overline{x}) \\ &\geqslant f_i(\overline{u}) + F(\overline{x}, \overline{u}; \alpha_i(\overline{x}, \overline{u})\nabla f_i(\overline{u})) + \rho_i d_i^2(\overline{x}, \overline{u}) \\ &\quad -\overline{\lambda}_i g_i(\overline{u}) + F(\overline{x}, \overline{u}; -\alpha_i(\overline{x}, \overline{u})\nabla_u \lambda_i g_i(\overline{u})) + \overline{\lambda}_i \rho_i d_i^2(\overline{x}, \overline{u}) \\ &\geqslant F(\overline{x}, \overline{u}; \alpha_i(\overline{x}, \overline{u})\nabla_u(f_i(\overline{u}) - \overline{\lambda}_i g_i(\overline{u}))) + \rho_i(1 + \overline{\lambda}_i) d_i^2(\overline{x}, \overline{u}). \end{aligned}$$

From $\overline{\tau}_i > 0$ and the sublinearity of $F$, we also know that

$$F(\overline{x}, \overline{u}; \overline{\tau}_i \nabla_u(f_i(\overline{u}) - \overline{\lambda}_i g_i(\overline{u}))) + \overline{\tau}_i \rho_i (1 + \overline{\lambda}_i) \frac{d_i^2(\overline{x}, \overline{u})}{\alpha_i(\overline{x}, \overline{u})} \leqslant 0,\ i = 1, 2, \cdots, p. \tag{17}$$

Furthermore, at least one of these inequalities holds strictly.

Adding the above inequalities together, we obtain

$$\sum_{i=1}^{p} F(\overline{x}, \overline{u}; \overline{\tau}_i \nabla_u(f_i(\overline{u}) - \overline{\lambda}_i g_i(\overline{u}))) + \sum_{i=1}^{p} \overline{\tau}_i \rho_i (1 + \overline{\lambda}_i) \frac{d_i^2(\overline{x}, \overline{u})}{\alpha_i(\overline{x}, \overline{u})} < 0.$$

By (1), this indicates that

$$F(\overline{x}, \overline{u}; \sum_{i=1}^{p} \overline{\tau}_i \nabla_u(f_i(\overline{u}) - \overline{\lambda}_i g_i(\overline{u}))) < -\sum_{i=1}^{p} \overline{\tau}_i \rho_i (1 + \overline{\lambda}_i) \frac{d_i^2(\overline{x}, \overline{u})}{\alpha_i(\overline{x}, \overline{u})}. \tag{18}$$

On the other hand, the $(F, \beta, \zeta_j, c_j)$-convexity of $h_j$ $(j = 1, 2, \cdots, m)$ yields

$$h_j(\overline{x}) - h_j(\overline{u}) \geqslant F(\overline{x}, \overline{u}; \beta(\overline{x}, \overline{u})\nabla h_j(\overline{u})) + \zeta_j c_j^2(\overline{x}, \overline{u}),\ j = 1, 2, \cdots, m.$$

By $\overline{v} \geqslant 0$, $\beta(\overline{x}, \overline{u}) > 0$ and the sublinearity of $F$, we have

$$\frac{\overline{v}_j(h_j(\overline{x}) - h_j(\overline{u}))}{\beta(\overline{x}, \overline{u})} \geqslant F(\overline{x}, \overline{u}; \overline{v}_j \nabla h_j(\overline{u})) + \overline{v}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})},\ j = 1, 2, \cdots, m.$$

Adding them together, we get

$$\sum_{j=1}^{m} \frac{\overline{v}_j(h_j(\overline{x}) - h_j(\overline{u}))}{\beta(\overline{x}, \overline{u})} \geqslant \sum_{j=1}^{m} F(\overline{x}, \overline{u}; \overline{v}_j \nabla h_j(\overline{u})) + \sum_{j=1}^{m} \overline{v}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})}.$$

Using (1), $\overline{v} \geqslant 0$, $\beta(\overline{x}, \overline{u}) > 0$, and the feasibility of $\overline{x}$ and $(\overline{u}, \overline{\lambda}, \overline{v})$, we obtain

$$F(\overline{x}, \overline{u}; \sum_{j=1}^{m} \overline{v}_j \nabla h_j(\overline{u})) + \sum_{j=1}^{m} \overline{v}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})} \leqslant 0,$$

i.e.,

$$F(\overline{x}, \overline{u}; \sum_{j=1}^{m} \overline{v}_j \nabla h_j(\overline{u})) \leqslant -\sum_{j=1}^{m} \overline{v}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})}. \tag{19}$$

Adding (18) and (19) together, and using the sublinearity of $F$ and the feasibility of $(\overline{u}, \overline{\lambda}, \overline{v})$, we have

$$\begin{aligned}
&-\sum_{i=1}^{p} \overline{\tau}_i \rho_i (1 + \overline{\lambda}_i) \frac{d_i^2(\overline{x}, \overline{u})}{\alpha_i(\overline{x}, \overline{u})} - \sum_{j=1}^{m} \overline{v}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})} \\
&> F(\overline{x}, \overline{u}; \sum_{i=1}^{p} \overline{\tau}_i \nabla_u (f_i(\overline{u}) - \overline{\lambda}_i g_i(\overline{u}))) + F(\overline{x}, \overline{u}; \sum_{j=1}^{m} \overline{v}_j \nabla h_j(\overline{u})) \\
&\geqslant F(\overline{x}, \overline{u}; 0) = 0.
\end{aligned}$$

This indicates

$$\sum_{i=1}^{p} \overline{\tau}_i \rho_i (1 + \overline{\lambda}_i) \frac{d_i^2(\overline{x}, \overline{u})}{\alpha_i(\overline{x}, \overline{u})} + \sum_{j=1}^{m} \overline{v}_j \zeta_j \frac{c_j^2(\overline{x}, \overline{u})}{\beta(\overline{x}, \overline{u})} < 0,$$

which contradicts (15). Therefore, the conclusion of the weak duality holds.

(II) It is clear that the second part of this corollary holds if the parameters $\alpha_i$, $d_i$ and $c_j$, are independent of $i$ or $j$, respectively. □

THEOREM 3.4. *(Strong Duality). Assume $\overline{x}$ is an efficient solution of (MFP), and the constraint qualification (GGCQ) holds at $\overline{x}$ (Ref. [14]). Then there exist $\overline{\tau} \in R_+^p$, $\overline{\lambda} \in R_+^p$, $\overline{v} \in R_+^m$ such that $(\overline{x}, \overline{\tau}, \overline{\lambda}, \overline{v})$ is a feasible solution of (MFD2) and*

$$\overline{\lambda} = \frac{f(\overline{x})}{g(\overline{x})}.$$

*Furthermore, if all assumptions in Theorem 3.3 are satisfied, then the corresponding $(\overline{x}, \overline{\tau}, \overline{\lambda}, \overline{v})$ is an efficient solution of (MFD2).*

*Proof.* Since $\overline{x}$ is an efficient solution of (MFP) and (GGCQ) holds at $\overline{x}$, there exist $\tau \in R_+^p$, $\tau > 0$, $\sum_{i=1}^{p} \tau_i = 1$, $v \in R_+^m$ such that

$$\begin{aligned}
&\sum_{i=1}^{p} \tau_i \nabla \frac{f_i(\overline{x})}{g_i(\overline{x})} + \sum_{j=1}^{m} v_j \nabla h_j(\overline{x}) = 0, \\
&v^T h(\overline{x}) = 0.
\end{aligned}$$

Denote

$$\overline{\lambda}_i = \frac{f_i(\overline{x})}{g_i(\overline{x})}, i = 1, 2, \cdots, p,$$

$$\overline{\tau}_i = \frac{\dfrac{\tau_i}{g_i(\overline{x})}}{\sum_{i=1}^{p} \dfrac{\tau_i}{g_i(\overline{x})}}, i = 1, 2, \cdots, p,$$

$$\overline{v}_j = \frac{v_j}{\sum_{i=1}^{p} \dfrac{\tau_i}{g_i(\overline{x})}}, j = 1, 2, \cdots, m.$$

Since

$$\nabla \frac{f_i(\overline{x})}{g_i(\overline{x})} = \frac{g_i(\overline{x}) \nabla f_i(\overline{x}) - f_i(\overline{x}) \nabla g_i(\overline{x})}{g_i^2(\overline{x})},$$

we can derive the following:

$$\begin{aligned}
&\sum_{i=1}^{p} \overline{\tau}_i \nabla_{\overline{x}} (f_i(\overline{x}) - \overline{\lambda}_i g_i(\overline{x})) + \sum_{j=1}^{m} \overline{v}_j \nabla h_j(\overline{x}) = 0, \\
&\overline{v}^T h(\overline{x}) = 0, \\
&f_i(\overline{x}) - \overline{\lambda}_i g_i(\overline{x}) = 0, \\
&\overline{\tau}, \overline{\lambda} \in R_+^p, \overline{\tau} > 0, \sum_{i=1}^{p} \overline{\tau}_i = 1, \overline{v} \in R_+^m,
\end{aligned}$$

i.e., $(\overline{x}, \overline{\tau}, \overline{\lambda}, \overline{v})$ is a feasible solution of (MFD2). Obviously, the corresponding objective function value of (MFD2) is equal to $\frac{f(\overline{x})}{g(\overline{x})}$. The proof of the last part is similar to that of Theorem 3.3. □

### 3.3. EXTENDED BECTOR TYPE DUAL

For a single-objective fractional programming problem in [4], Bector used the positivity of the denominator to transform the inequality constraints and add them to the objective by Lagrangian mulitipliers for establishing a kind of dual. Since the denominators in (MFP) need not be the same, we use the equivalent form $(\overline{MFP})$ of (MFP) to establish the following dual, which is called the extended Bector type dual of (MFP):

$$\text{(MFD3)}\ \max\ \left(\frac{G_1(u)f_1(u)+v_{M_0}^T h_{M_0}(u)}{G(u)},\cdots,\frac{G_p(u)f_p(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}\right)^T,$$

$$\begin{aligned}\text{s.t.}\ & \sum_{i=1}^{p}\tau_i\nabla_u\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}+\sum_{k=1}^{q}\nabla_u v_{M_k}^T h_{M_k}(u)=0,\\ & v_{M_k}^T h_{M_k}(u)\geqslant 0, \qquad k=1,2,\cdots,q,\\ & G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)\geqslant 0, \qquad i=1,2,\cdots,p,\\ & \sum_{i=1}^{p}\tau_i=1,\ \tau=(\tau_1,\tau_2,\cdots,\tau_p)^T\in R_+^p,\ \tau>0,\\ & u\in X,\ v_{M_k}\in R_+^{|M_k|}, \qquad k=0,1,2,\cdots,q.\end{aligned}$$

THEOREM 3.5. *(Weak Duality) Let $x$ be a feasible solution of (MFP) and $(u,\tau,v)$ be a feasible solution of (MFD3). Assume that $-G$ is $(F,\alpha,\rho,d)$-convex at $u$, $G_i f_i$ ($i=1,2,\cdots,p$) is $(F,\alpha,\rho_i,d)$-convex at $u$ and $h_j$ ($j=1,2,\cdots,m$) is $(F,\alpha,\zeta_j,d)$-convex at $u$. If $\rho\geqslant\max_{1\leqslant i\leqslant p}\rho_i$ and the following inequality holds:*

$$\begin{aligned}&\textstyle\sum_{i=1}^{p}\tau_i\rho_i\left(1+\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}\right)\\ &+\textstyle\sum_{j\in M_0}v_j\zeta_j+G(u)\sum_{k=1}^{q}\sum_{j\in M_k}v_j\zeta_j\geqslant 0,\end{aligned}\tag{20}$$

*then we have*

$$\frac{f(x)}{g(x)}\nprec\frac{\overline{G}(u)f(u)+v_{M_0}^T h_{M_0}(u)\,e}{G(u)},$$

*where $\overline{G}(u)=diag\{G_1(u),\cdots,G_p(u)\}$ and each component in $e\in R^p$ is equal to 1.*

*Proof.* Suppose to the contrary that

$$\frac{f(x)}{g(x)}\prec\frac{\overline{G}(u)f(u)+v_{M_0}^T h_{M_0}(u)\,e}{G(u)}.$$

For any $i$, $1\leqslant i\leqslant p$, the inequality

$$\frac{f_i(x)}{g_i(x)}\leqslant\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}$$

is equivalent to the following:

$$\frac{G_i(x)f_i(x)}{G(x)}-\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}\leqslant 0,$$

i.e.,

$$G_i(x)f_i(x)G(u)-(G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u))G(x)\leqslant 0.$$

Denote

$$\Phi_i(x) = G_i(x) f_i(x) G(u) - (G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u)) G(x).$$

Then, by hypothesis, we know that

$$\Phi_i(x) \leqslant 0, i = 1, 2, \cdots, p, \tag{21}$$

and at least one of these inequalities holds strictly.

Since $\Phi_i(u) = -v_{M_0}^T h_{M_0}(u) G(u)$, we have

$$\begin{aligned}\Phi_i(x) &= \Phi_i(x) - \Phi_i(u) - v_{M_0}^T h_{M_0}(u) G(u) \\ &= G(u)(G_i(x) f_i(x) - G_i(u) f_i(u)) + (G_i(u) f_i(u) \\ &\quad + v_{M_0}^T h_{M_0}(u))(-G(x) + G(u)) - v_{M_0}^T h_{M_0}(u) G(u).\end{aligned}$$

Note that, for $i = 1, 2, \cdots, p$, $-G(x)$ is also $(F, \alpha, \rho_i, d)$-convex at $u$. By the $(F, \alpha, \rho_i, d)$-convexity of $G_i(x) f_i(x)$ and $-G(x)$, $G(u) > 0$, and $G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u) \geqslant 0$, we get

$$\begin{aligned}\Phi_i(x) &\geqslant G(u)(F(x, u; \alpha(x, u) \nabla(G_i(u) f_i(u))) + \rho_i d^2(x, u)) \\ &\quad + (G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u))(F(x, u; -\alpha(x, u) \nabla G(u)) \\ &\quad + \rho_i d^2(x, u)) - v_{M_0}^T h_{M_0}(u) G(u).\end{aligned}$$

Furthermore, using the sublinearity of $F$ and $\alpha(x, u) > 0$, we obtain

$$\begin{aligned}\Phi_i(x) &\geqslant \alpha(x, u) F(x, u; G(u) \nabla(G_i(u) f_i(u)) + G(u) \rho_i d^2(x, u) \\ &\quad + \alpha(x, u) F(x, u; -(G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u)) \nabla G(u)) \\ &\quad + (G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u)) \rho_i d^2(x, u) - v_{M_0}^T h_{M_0}(u) G(u).\end{aligned}$$

Adding the term $\alpha(x, u) F(x, u; G(u) \nabla_u (v_{M_0}^T h_{M_0}(u)))$ and its negative to the right-hand side of the above inequality and using the sublinearity of $F$ again, we have

$$\begin{aligned}\Phi_i(x) &\geqslant \alpha(x, u) F(x, u; G(u) \nabla(G_i(u) f_i(u))) \\ &\quad + \alpha(x, u) F(x, u; G(u) \nabla_u (v_{M_0}^T h_{M_0}(u))) \\ &\quad - \alpha(x, u) F(x, u; G(u) \nabla_u (v_{M_0}^T h_{M_0}(u))) + G(u) \rho_i d^2(x, u) \\ &\quad + \alpha(x, u) F(x, u; -(G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u)) \nabla G(u)) \\ &\quad + (G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u)) \rho_i d^2(x, u) - v_{M_0}^T h_{M_0}(u) G(u) \\ &\geqslant \alpha(x, u) F\bigg(x, u; G(u) \nabla_u (G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u)) \\ &\quad - (G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u)) \nabla G(u)\bigg) \\ &\quad - \alpha(x, u) F(x, u; G(u) \nabla_u (v_{M_0}^T h_{M_0}(u))) + G(u) \rho_i d^2(x, u) \\ &\quad + (G_i(u) f_i(u) + v_{M_0}^T h_{M_0}(u)) \rho_i d^2(x, u) - v_{M_0}^T h_{M_0}(u) G(u).\end{aligned}$$

The last inequality is equivalent to the following:

$$\begin{aligned}\Phi_i(x) \;\geqslant\; & \alpha(x,u)G^2(u)F\left(x,u;\nabla_u\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}\right)\\ & -\alpha(x,u)F(x,u;G(u)\nabla_u(v_{M_0}^T h_{M_0}(u)))+G(u)\rho_i d^2(x,u)\\ & +(G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u))\rho_i d^2(x,u)-v_{M_0}^T h_{M_0}(u)G(u).\end{aligned}$$

Let us multiply the above inequality by $\tau_i$ for $i=1,\cdots,p$, respectively, and add them together. Since at least one of inequalities in (21) holds strictly, $\tau_i>0$ and $\sum_{i=1}^{p}\tau_i=1$, by using the sublinearity of $F$, we can obtain

$$\begin{aligned}0 \;>\; & \sum_{i=1}^{p}\tau_i\Phi_i(x)\\ \geqslant\; & \alpha(x,u)G^2(u)F\left(x,u;\sum_{i=1}^{p}\tau_i\nabla_u\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}\right)\\ & -\alpha(x,u)F(x,u;G(u)\nabla_u(v_{M_0}^T h_{M_0}(u)))+\sum_{i=1}^{p}\tau_i G(u)\rho_i d^2(x,u)\\ & +\sum_{i=1}^{p}\tau_i(G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u))\rho_i d^2(x,u)-v_{M_0}^T h_{M_0}(u)G(u).\end{aligned}$$

Note that $(u,\tau,v)$ is dual feasible, and so it follows that

$$\sum_{i=1}^{p}\tau_i\nabla_u\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}+\sum_{k=1}^{q}\nabla_u v_{M_k}^T h_{M_k}(u)=0.$$

Hence, we have

$$\begin{aligned}0 \;>\; & \sum_{i=1}^{p}\tau_i\Phi_i(x)\\ \geqslant\; & \alpha(x,u)G^2(u)F(x,u;-\sum_{k=1}^{q}\nabla_u(v_{M_k}^T h_{M_k}(u)))\\ & -\alpha(x,u)F(x,u;G(u)\nabla_u(v_{M_0}^T h_{M_0}(u)))+\sum_{i=1}^{p}\tau_i G(u)\rho_i d^2(x,u)\\ & +\sum_{i=1}^{p}\tau_i(G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u))\rho_i d^2(x,u)-v_{M_0}^T h_{M_0}(u)G(u).\end{aligned} \tag{22}$$

On the other hand, by the $(F,\alpha,\zeta_j,d)$-convexity of $h_j$, $j\in M_0$, we have

$$h_j(x)-h_j(u)\geqslant F(x,u;\alpha(x,u)\nabla h_j(u))+\zeta_j d^2(x,u).$$

After multiplying the above inequality by $v_j$ for each $j\in M_0$, we add them together. By the feasibility of $x$ and $(u,\tau,v)$, $G(u)>0$, $v_j\geqslant 0$, and the sublinearity of $F$, we get

$$\begin{aligned} -G(u)\sum_{j\in M_0} v_j h_j(u)) &\geqslant G(u)(\sum_{j\in M_0} v_j h_j(x) - \sum_{j\in M_0} v_j h_j(u)) \\ &\geqslant G(u)\sum_{j\in M_0} v_j F(x,u;\alpha(x,u)\nabla h_j(u)) \\ &\quad + G(u)\sum_{j\in M_0} v_j \zeta_j d^2(x,u) \\ &\geqslant G(u)F(x,u;\alpha(x,u)\sum_{j\in M_0} v_j \nabla h_j(u)) \\ &\quad + G(u)\sum_{j\in M_0} v_j \zeta_j d^2(x,u), \end{aligned}$$

that is,

$$\begin{aligned} &-G(u)v_{M_0}^T h_{M_0}(u) - G(u)F(x,u;\alpha(x,u)\nabla_u(v_{M_0}^T h_{M_0}(u))) \\ &\geqslant G(u)\sum_{j\in M_0} v_j \zeta_j d^2(x,u). \end{aligned}$$

Hence, by (22), we obtain

$$\begin{aligned} 0 &> \sum_{i=1}^{p} \tau_i \Phi(x) \\ &\geqslant \alpha(x,u)G^2(u)F(x,u;-\sum_{k=1}^{q}\nabla_u(v_{M_k}^T h_{M_k}(u))) \\ &\quad + \sum_{i=1}^{p}\tau_i G(u)\rho_i d^2(x,u) + G(u)\sum_{j\in M_0} v_j \zeta_j d^2(x,u) \\ &\quad + \sum_{i=1}^{p}\tau_i (G_i(u)f_i(u) + v_{M_0}^T h_{M_0}(u))\rho_i d^2(x,u). \end{aligned} \tag{23}$$

For $k = 1, 2, \cdots, q$, $j \in M_k$, by the $(F, \alpha, \zeta_j, d)$-convexity of $h_j$, the feasibility of $x$ and $(u, \tau, v)$, we obtain

$$\begin{aligned} 0 &\geqslant \sum_{k=1}^{q} v_{M_k}^T (h_{M_k}(x) - h_{M_k}(u)) \\ &\geqslant \sum_{k=1}^{q}\sum_{j\in M_k} v_j (F(x,u;\alpha(x,u)\nabla h_j(u)) + \zeta_j d^2(x,u)) \\ &\geqslant F(x,u;\alpha(x,u)\sum_{k=1}^{q}\nabla_u(v_{M_k}^T h_{M_k}(u))) + \sum_{k=1}^{q}\sum_{j\in M_k} v_j \zeta_j d^2(x,u). \end{aligned} \tag{24}$$

Multiplying (24) by $G^2(u) > 0$ and adding it to (23), we have

$$\begin{aligned} 0 \ > \ & \alpha(x,u)G^2(u)F(x,u;-\sum_{k=1}^{q}\nabla_u(v_{M_k}^T h_{M_k}(u))) \\ & +\sum_{i=1}^{p}\tau_i G(u)\rho_i d^2(x,u) + G(u)\sum_{j\in M_0} v_j\zeta_j d^2(x,u) \\ & +\sum_{i=1}^{p}\tau_i(G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u))\rho_i d^2(x,u) \\ & +\alpha(x,u)G^2(u)F(x,u;\sum_{k=1}^{q}\nabla_u(v_{M_k}^T h_{M_k}(u))) \\ & +G^2(u)\sum_{k=1}^{q}\sum_{j\in M_k} v_j\zeta_j d^2(x,u). \end{aligned}$$

Since $G(u) > 0$, dividing the two sides of the above inequality by $G(u)$ and using the sublinearity of $F$, we obtain

$$\begin{aligned} 0 > \ & \Bigg(\sum_{i=1}^{p}\tau_i\rho_i(1+\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}) + \sum_{j\in M_0} v_j\zeta_j \\ & +G(u)\sum_{k=1}^{q}\sum_{j\in M_k} v_j\zeta_j\Bigg)d^2(x,u), \end{aligned}$$

which contradicts (20). Hence, the conclusion of Theorem 3.5 holds. □

THEOREM 3.6. *(Strong Duality) Assume that $\overline{x}$ is an efficient solution of (MFP) and the constraint qualification (GGCQ) holds at $\overline{x}$ (Ref. [14]). Then there exists $(\overline{\tau},\overline{v})$ such that $(\overline{x},\overline{\tau},\overline{v})$ is a feasible solution of (MFD3), and the objective function values of (MFP) and (MFD3) at $\overline{x}$ and $(\overline{x},\overline{\tau},\overline{v})$, respectively, are equal. If the assumptions and conditions in Theorem 3.5 are also satisfied, then $(\overline{x},\overline{\tau},\overline{v})$ is an efficient solution of (MFD3).*

*Proof.* Since $\overline{x}$ is an efficient solution of (MFP) and (GGCQ) holds at $\overline{x}$, there exists $\overline{\tau}\in R_+^p$, $\overline{\tau} > 0$, $\sum_{i=1}^{p}\overline{\tau}_i = 1$, $\overline{v}\in R_+^m$ such that

$$\sum_{i=1}^{p}\overline{\tau}_i\nabla\frac{f_i(\overline{x})}{g_i(\overline{x})} + \sum_{j=1}^{m}\overline{v}_j\nabla h_j(\overline{x}) = 0, \tag{25}$$

$$\overline{v}^T h(\overline{x}) = 0. \tag{26}$$

Since $\overline{x}$ is also feasible for (MFP), $h_j(\overline{x}) \leqslant 0$ for $j = 1,2,\cdots,m$. Hence, by (26) and $\overline{v}_j \geqslant 0$, we have

$$\begin{aligned} & \overline{v}_{M_0}^T h_{M_0}(\overline{x}) = 0, \\ & \overline{v}_{M_k}^T h_{M_k}(\overline{x}) = 0, k = 1,2,\cdots,q. \end{aligned}$$

It is easy to verify that

$$\sum_{j=1}^{m} \overline{v}_j \nabla h_j(\overline{x}) = \sum_{k=0}^{p} \nabla_{\overline{x}}(\overline{v}_{M_k}^T h_{M_k}(\overline{x})),$$

and, for $i = 1, \cdots, p$,

$$\nabla \frac{f_i(\overline{x})}{g_i(\overline{x})} = \nabla_{\overline{x}} \frac{G_i(\overline{x}) f_i(\overline{x})}{G(\overline{x})}.$$

From (25) and $\sum_{i=1}^{p} \overline{\tau}_i = 1$, we have

$$\begin{aligned}
&\sum_{i=1}^{p} \overline{\tau}_i \nabla \frac{f_i(\overline{x})}{g_i(\overline{x})} + \sum_{j=1}^{m} \overline{v}_j \nabla h_j(\overline{x}) \\
&= \sum_{i=1}^{p} \overline{\tau}_i \nabla_{\overline{x}} \frac{G_i(\overline{x}) f_i(\overline{x})}{G(\overline{x})} + \nabla_{\overline{x}}(\overline{v}_{M_0}^T h_{M_0}(\overline{x})) + \sum_{k=1}^{q} \nabla_{\overline{x}}(\overline{v}_{M_k}^T h_{M_k}(\overline{x})) \\
&= \sum_{i=1}^{p} \overline{\tau}_i \nabla_{\overline{x}} \frac{G_i(\overline{x}) f_i(\overline{x}) + G(\overline{x}) \overline{v}_{M_0}^T h_{M_0}(\overline{x})}{G(\overline{x})} + \sum_{k=1}^{q} \nabla_{\overline{x}}(\overline{v}_{M_k}^T h_{M_k}(\overline{x})) \\
&= 0.
\end{aligned}$$

The above equations imply that

$$\begin{aligned}
&\sum_{i=1}^{p} \overline{\tau}_i \nabla_{\overline{x}} \frac{G_i(\overline{x}) f_i(\overline{x}) + G(\overline{x}) \overline{v}_{M_0}^T h_{M_0}(\overline{x})}{G(\overline{x})} + \sum_{k=1}^{q} \nabla_{\overline{x}}(\overline{v}_{M_k}^T h_{M_k}(\overline{x})) = 0, \\
&G(\overline{x}) \overline{v}_{M_k}^T h_{M_k}(\overline{x}) = 0, k = 1, 2, \cdots, q, \\
&G_i(\overline{x}) f_i(\overline{x}) + G(\overline{x}) \overline{v}_{M_0}^T h_{M_0}(\overline{x}) \geqslant 0, \\
&\overline{v}_{M_k} G(\overline{x}) \in R_+^{|M_k|}, k = 0, 1, 2, \cdots, q.
\end{aligned}$$

This indicates that $(\overline{x}, \overline{\tau}, \overline{v} G(\overline{x}))$ is also a feasible solution of (MFD3). Since $\overline{v}_{M_0}^T h_{M_0}(\overline{x}) = 0$, the values of the corresponding objective functions of (MFP) and (MFD3) are equal. Obviously, if the assumptions about the generalized convexity of the related functions and other conditions in Theorem 3.5 are also satisfied, then $(\overline{x}, \overline{\tau}, \overline{v} G(\overline{x}))$ is an efficient solution of (MFD3). □

## 4. Concluding Remarks

In this paper, a unified formulation of the generalized convexity defined in [12] is adopted, which includes many other generalized convexity concepts in optimization theory as special cases. Our concept of generalized convexity is suitable to analyze the efficiency conditions and duality of multiobjective fractional programming problems. Efficiency conditions and duality for a class of multiobjective fractional programming problems are presented. We extend the methods, which were adopted for single-objective fractional programming problems in [10, 12, 21], to the case with multiple fractional objectives. We also present the extended Bector type dual by using an equivalent formulation of the primal problem. Note that we only consider (MFP) from a viewpoint of the efficient solution in this paper. The methods used here can be extended to the study of (MFP) from a viewpoint of the weak efficient solution.

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